STATISTICS and PROBABILITY

LECTURE: CONFIDENCE INTERVAL

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objectives of this lecture

Confidence Interval

After carefully following this lecture, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the t distribution method.
2. Construct confidence intervals on the variance and standard deviation of a normal distribution.
3. Construct confidence intervals on a population proportion.
4. Use a general method for constructing an approximate confidence interval on a parameter.
5. Construct prediction intervals for a future observation.
6. Construct a tolerance interval for a normal population.
7. Explain the three types of interval estimates: Confidence intervals, prediction intervals, and tolerance intervals.
In the previous lecture we illustrated how a parameter can be estimated from sample data. However, it is important to understand how good is the estimate obtained.

Bounds that represent an interval of plausible values for a parameter are an example of an interval estimate.

Three types of intervals will be presented:

- Confidence intervals
- Prediction intervals
- Tolerance intervals
$\bar{X}$ is normally distributed with mean $\mu$ and variance $\sigma^2/n$. We may standardize $\bar{X}$ by subtracting the mean and dividing by the standard deviation, which results in the variable

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad (8-1)$$

The random variable $Z$ has a standard normal distribution.
A confidence interval estimate for $\mu$ is an interval of the form $l \leq \mu \leq u$, where the endpoints $l$ and $u$ are computed from the sample data. Because different samples will produce different values of $l$ and $u$, these end-points are values of random variables $L$ and $U$, respectively. Suppose that we can determine values of $L$ and $U$ such that the following probability statement is true:

$$P\{L \leq \mu \leq U\} = 1 - \alpha$$

(8-2)

where $0 \leq \alpha \leq 1$. There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of $\mu$. Once we have selected the sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and computed $l$ and $u$, the resulting confidence interval for $\mu$ is

$$l \leq \mu \leq u$$

(8-3)
The endpoints or bounds $l$ and $u$ are called lower- and upper-confidence limits, respectively.

Since $Z$ follows a standard normal distribution, we can write:

$$P\left\{-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right\} = 1 - \alpha$$

Now manipulate the quantities inside the brackets by (1) multiplying through by $\sigma/\sqrt{n}$, (2) subtracting $\bar{X}$ from each term, and (3) multiplying through by $-1$. This results in

$$P\left\{\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha$$

(8-4)
Definition

If $\bar{x}$ is the sample mean of a random sample of size $n$ from a normal population with known variance $\sigma^2$, a $100(1 - \alpha)$% CI on $\mu$ is given by

$$\bar{x} - z_{\alpha/2} \sigma / \sqrt{n} \leq \mu \leq \bar{x} + z_{\alpha/2} \sigma / \sqrt{n} \tag{8-5}$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.
Example 1

Confidence Interval

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature.

Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with \( \sigma = 1 \)J.

We want to find a 95% CI for \( \mu \) the mean impact energy.
Example

Confidence Interval

SOLUTION:
Confidence Level and Precision of Error

The length of a confidence interval is a measure of the precision of estimation.

Figure: Error in estimating $\mu$ with $\bar{x}$
If \( \bar{x} \) is used as an estimate of \( \mu \), we can be \( 100(1 - \alpha)\% \) confident that the error \( |\bar{x} - \mu| \) will not exceed a specified amount \( E \) when the sample size is

\[
n = \left( \frac{z_{\alpha/2} \sigma}{E} \right)^2 \quad (8-6)
\]
Example 2

Confidence Interval

To illustrate the use of this procedure, consider the Example 1 in which CVN test is described. 

How large should the sample size be if in estimating the mean impact energy there is to be a 95% CI that the error will be less than 0.5J.
A Large-Sample Confidence Interval for $\mu$

When $n$ is large, the quantity

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\bar{x} - z_{\alpha/2} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2} \frac{s}{\sqrt{n}}$$

(8-11)

is a large sample confidence interval for $\mu$, with confidence level of approximately $100(1 - \alpha)\%$. 
Example 3: Confidence Interval

The sample mean and sample standard deviation for AGNO of a random sample of 100 freshman at Ataturk University are 2.5 and 0.2, respectively.

Find a 99% CI for the mean $\mu$ of AGNO for the entire freshman class.
Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2$. The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

(8-13)

has a $t$ distribution with $n - 1$ degrees of freedom.
Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

The $t$ distribution

Probability density functions of several $t$ distributions.
Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

**The 𝑡 distribution**

*Figure:* Percentage points of the 𝑡 distribution.
The *t* Confidence Interval on $\mu$

If $\bar{x}$ and $s$ are the mean and standard deviation of a random sample from a normal distribution with unknown variance $\sigma^2$, a $100(1 - \alpha)\%$ confidence interval on $\mu$ is given by

$$\bar{x} - t_{\alpha/2, n-1} s / \sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2, n-1} s / \sqrt{n}$$  \hspace{1cm} (8-16)

where $t_{\alpha/2, n-1}$ is the upper $100\alpha/2$ percentage point of the *t* distribution with $n - 1$ degrees of freedom.
Example 4: Confidence Interval


The load at specimen failure is as follows (in megapascals);

<table>
<thead>
<tr>
<th>19.8</th>
<th>10.1</th>
<th>14.9</th>
<th>7.5</th>
<th>15.4</th>
<th>15.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>18.5</td>
<td>7.9</td>
<td>12.7</td>
<td>11.9</td>
<td>11.4</td>
</tr>
<tr>
<td>11.4</td>
<td>14.1</td>
<td>17.6</td>
<td>16.7</td>
<td>15.8</td>
<td></td>
</tr>
<tr>
<td>19.5</td>
<td>8.8</td>
<td>13.6</td>
<td>11.9</td>
<td>11.4</td>
<td></td>
</tr>
</tbody>
</table>

Find a 95% CI for the mean $\mu$ of the load causing the failure of the adhesion
Let \( X_1, X_2, \ldots, X_n \) be a random sample from a normal distribution with mean \( \mu \) and variance \( \sigma^2 \), and let \( S^2 \) be the sample variance. Then the random variable

\[
X^2 = \frac{(n - 1) S^2}{\sigma^2}
\]  

(8-17)

has a chi-square (\( \chi^2 \)) distribution with \( n - 1 \) degrees of freedom.
Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Figure: Probability density functions of several $\chi^2$ distributions.
If $s^2$ is the sample variance from a random sample of $n$ observations from a normal distribution with unknown variance $\sigma^2$, then a $100(1 - \alpha)\%$ confidence interval on $\sigma^2$ is

\[
\frac{(n - 1)s^2}{\chi^2_{\alpha/2, n-1}} \leq \sigma^2 \leq \frac{(n - 1)s^2}{\chi^2_{1-\alpha/2, n-1}}
\]  \hspace{1cm} (8-19)

where $\chi^2_{\alpha/2, n-1}$ and $\chi^2_{1-\alpha/2, n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n - 1$ degrees of freedom, respectively. A confidence interval for $\sigma$ has lower and upper limits that are the square roots of the corresponding limits in Equation 8-19.
Figure 8-9  Percentage point of the $\chi^2$ distribution. (a) The percentage point $\chi^2_{\alpha,k}$. (b) The upper percentage point $\chi^2_{0.05,10} = 18.31$ and the lower percentage point $\chi^2_{0.95,10} = 3.94$. 
Example 5: \hspace{1cm} \textit{Confidence Interval}

A rivet is to be inserted into a hole. A random sample of \( n = 15 \) parts is selected, and the hole diameter is measured.

The sample standard deviation of the hole diameter measurements is \( s = 0.008 \) millimeters. Construct a 99% lower confidence bound for the variance.
If $n$ is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1 - p)}} = \frac{\hat{P} - p}{\sqrt{p(1 - p)/n}}$$

is approximately standard normal.

The quantity $\sqrt{p(1 - p)/n}$ is called the standard error of the point estimator $\hat{P}$.
If $\hat{p}$ is the proportion of observations in a random sample of size $n$ that belongs to a class of interest, an approximate 100$(1 - \alpha)\%$ confidence interval on the proportion $p$ of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \leq p \leq \hat{p} + z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$  \hspace{1cm} (8-23)

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.
Example 6: Confidence Interval

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow.

Find a 95% CI for the bearings that exceeds the roughness specification.
Hypothesis Testing.....